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On the exponential diophantine equations of the form $a^x - b^y \cdot c^z = \underline{+}1$ with a, b, c, prime numbers

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Abstract

In this short paper we obtain by elementary methods some general results for some exponential diophantine equations of the form

$$a^x - b^y \cdot c^z = \underline{+}1, \tag{1}$$

with a, b, c prime natural numbers, and x, y, z natural numbers.

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1 Introduction

Using only the congruences with a single modulus, D.Acu, L.J.Alex, L.S.Chen, Y.B.Wang obtain all the solutions for some diophantine equations of the form (1) such are:

$$7^x - 2^y \cdot 3^z = \underline{+}1, \qquad (see\ [3]);$$

$$5^x - 2^y \cdot 7^z = \underline{+}1,\ 7^x - 2^y \cdot 5^z = \underline{+}1, \quad (see\ [4]);$$

$$2^x - 3^y \cdot 17^z = \underline{+}1,\ 2^x - 3^y \cdot 19^z = \underline{+}1, \quad (see\ [1]);$$

$$3^x - 2^y \cdot 5^z = \underline{+}1,\ 5^x - 2^y \cdot 3^z = \underline{+}1 \quad (see\ [5]).$$

Theorem 1 [2] *The only solutions of the equations* $3^x - 2^y \cdot 11^z = 1$ *in nonnegative integers are (1, 1, 0), (2, 3, 0) and (5, 1, 2).*

Theorem 2 [6] *The equation* $a^x - p_1^{y_1} p_2^{y_2} ... p_k^{y_k} = \underline{+}1$ *where* a *is a positive integer with* $a > 1$ *and* $p_1, p_2, ...p_k$ *are distinct primes with* $gcd(a, p_1 p_2 ... p_k) = 1$*, has only finitely many positive integer solutions* $(x, y_1, ..., y_k)$.

2 New results

Theorem 3 *If* a, b, c *are distinct prime numbers and not equal with 2, then the equations* $a^x - b^y \cdot c^z = \underline{+}1$ *have no nonnegative integer solutions.*

Proof. The equation $a^x - b^y \cdot c^z = 1$ is equivalent with

$$a^x - 1 = b^y \cdot c^z \tag{2}$$

For b, c prime numbers not equal with 2 we obtain

$$b^y \cdot c^z \not\equiv 0\ (mod\ 2) \tag{3}$$

Because a prime number not equal with 2, is an odd number we obtain:

$$a^x = (2q+1)^x \equiv 1 \ (mod 2) \tag{4}$$

$$a^x - 1 \equiv 0 \ (mod\ 2)$$

Using (2), (3), (4) we find a contradiction. We conclude that the equation $a^x - b^y \cdot c^z = 1$ have no nonnegative integer solutions.

The equation $a^x - b^y \cdot c^z = -1$ is equivalent with

$$a^x + 1 = b^y \cdot c^z \tag{5}$$

In a similar way we obtain $b^y \cdot c^z \not\equiv 0 (mod 2)$ and $a^x + 1 \equiv 0$ (mod 2). Using (5) we find a contradiction. We conclude that the equation $a^x - b^y \cdot c^z = -1$ has no nonnegative integer solutions.

Theorem 4 *If a is a positive integer with the unit digit 1, b, c distinct prime numbers and not equal with 5, then the equations* $a^x - b^y \cdot c^z = 1$ *have no nonnegative integer solutions.*

Proof. The equation $a^x - b^y \cdot c^z = 1$ is equivalent with

$$a^x - 1 = b^y \cdot c^z \tag{6}$$

For b, c distinct prime numbers and not equal with 5 we obtain

$$b^y \cdot c^z \not\equiv 0 \ (mod\ 5) \tag{7}$$

For $x = 0$ we obtain

$$a^x - 1 \equiv 0 \ (mod\ 5) \tag{8}$$

For $x \geq 1$ we obtain a^x is a positive integer with the unit digit 1. Now, we obtain $a^x - 1$ is a positive integer with the unit digit 0. We conclude that

$$a^x - 1 \equiv 0 \ (mod\ 5) \tag{9}$$

Using (6), (7), (8), (9) we find a contradiction. We conclude that the equation $a^x - b^y \cdot c^z = 1$ has no nonnegative integer solutions.

Theorem 5 *If a is a nonnegative integer with $a \equiv 1 \pmod{3}$, b, c are nonnegative integers with $b \cdot c$ not equivalent with 0 (mod 3) then the equation $a^x - b^y \cdot c^z = 1$ have no nonnegative integer solutions.*

Proof. Using for the equation the form $a^x - 1 = b^y \cdot c^z$ we obtain from bc not equivalent with 0 (mod 3) that $b^y \cdot c^z$ not equivalent with 0 (mod 3) and from $a \equiv 1$ (mod 3), we obtain $a^x - 1 \equiv 0$ (mod 3). Now we find a contradiction. We conclude by saying that the equation $a^x - b^y \cdot c^z = 1$ has no nonnegative integer solutions.

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